satisfies, as in the previous example, the conditions formulated above which assert the instability (which is obvious in the present case) of equilibrium.

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ON THE OSCILLATIONS OF A SYSTEM OF COUPLED OSCILLATORS WITH ONE THIRD-ORDER RESONANCE

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The case when there is one resonance relation $\beta_1 = 2\beta_2$ between the frequencies of oscillators was studied in [1, 2]. We consider the possible case of a third-order resonance in the oscillations in a Hamiltonian system of nonlinearly coupled oscillators when there is one resonance relation of the form $\beta_1 + \beta_2 = \beta_3$ [1] between the frequencies of three oscillators. This problem was studied by using the method of secular perturbations in [6].

1. Statement of the problem. We consider a Hamiltonian system of nonlinearly coupled oscillators with the Hamiltonians

$$H(p, q) = H_2(p, q) + H_3(p, q) + \ldots + H_i(p, q) + \ldots \tag{1.1}$$

$$p = (p_1, \ldots, p_n), \quad q = (q_1, \ldots, q_n)$$

$$H_2(p, q) = \frac{1}{2} \sum_{\nu=1}^{n} \beta_\nu (q_\nu^2 + p_\nu^2) \qquad (\beta_\nu > 0) \tag{1.2}$$

Here $\pm i\beta_\nu$ are the eigenvalues of the linearized system; $H_i(p, q)$ are homogeneous polynomials of degree i. The quantities $\beta_\nu > 0$ corresponding to the frequencies of the "uncoupled" oscillators, i.e., to the case when all $H_i(p, q) = 0$ $(i \geqslant 3)$ in (1.1) are simply called frequencies in what follows.

Let there exist a relation

$$k_1\beta_1 + k_2\beta_2 + \ldots + k_n\beta_n = 0 \tag{1.3}$$

where the k_i are integers. Then we say that resonance occurs. The vector $k = (k_1, \ldots, k_n)$ is called the resonance vector, while the number $k = |k_1| + \ldots + |k_n|$ is called the order of the resonance. We consider a system of n oscillators in the case when there is only one linearly independent resonance relation (1.3) between the frequencies of the oscillators, which (for an appropriate numbering of the oscillators) we write:

$$\beta_1 + \beta_2 = \beta_3 \tag{1.4}$$

In this case, according to the theorem [3] on the possibility of reducing a Hamiltonian system in the case of resonance (1.3) to the simplest, so-called "normal" form, the Hamiltonian system (1.1), (1.2) can be reduced by a canonic polynomial change of variables ($p, q \to \xi, \eta$) to the form

$$H = \sum_{\nu=1}^{n} \beta_\nu \rho_\nu + 2A\sqrt{\rho_1\rho_2\rho_3}\cos\psi + R(\rho, \varphi) \tag{1.5}$$

$$\xi_\nu = \sqrt{2\rho_\nu}\sin\varphi_\nu, \qquad \eta_\nu = \sqrt{2\rho_\nu}\cos\varphi_\nu \tag{1.6}$$

Here ρ_ν, φ_ν are canonic polar coordinates; ψ is the "resonance phase":

$$\psi = \varphi_1 + \varphi_2 - \varphi_3 \tag{1.7}$$

$R(\rho, \varphi)$ is of no less than second degree in the variable ρ

The Hamiltonian $\Gamma = H - R(\rho, \varphi)$ differing from H by terms of no less than second order in the variable ρ, i.e., by terms of higher than third order in the original variable, coincides with the accuracy indicated with the "normal form" of the Hamiltonian [3, 4]. In what follows we consider a model system with the Hamiltonian

$$\Gamma = \sum_{\nu=1}^{n} \beta_\nu \rho_\nu + 2A\sqrt{\rho_1\rho_2\rho_3}\cos\psi \tag{1.8}$$

In the variables ρ, φ of the Hamiltonian the form of the system of equations of motion is

$$\frac{d\rho_\nu}{dt} = \frac{\partial\Gamma}{\partial\varphi_\nu}, \quad \frac{d\varphi_\nu}{dt} = -\frac{\partial\Gamma}{\partial\rho_\nu} \tag{1.9}$$

Let us assume that the constant A in (1.8) is not zero. We remark that since a system with Hamiltonian (1.8) is, to within the accuracy indicated, a model system for "all" (*) systems of n coupled oscillators with Hamiltonian (1.1), (1.2) and resonance relation (1.4) to be considered by us, a study of it enables us to give a motion portrait for "all" such systems in one.

System (1.9) with Hamiltonian (1.8) possesses the following integrals [3, 1]

$$J_2 = \rho_2 - \frac{k_2}{k_1}\rho_1 = \rho_2 - \rho_1, \qquad J_3 = \rho_3 - \frac{k_3}{k_1}\rho_1 = \rho_3 + \rho_1 \tag{1.10}$$

$$J_4 = \rho_4, \ J_5 = \rho_5, \ldots, J_n = \rho_n$$

Here k_α are the components of the n-dimensional resonance vector k (1, 1, -1, 0, ..., 0). For system (1.9) there exists further the integral

$$F = 2A\sqrt{\rho_1\rho_2\rho_3}\cos\psi \tag{1.11}$$

The equation for the phases φ_j have the form

$$\frac{d\varphi_j}{dt} = -\beta_j \qquad (j = 4, 5, \ldots, n) \tag{1.12}$$

*) For some systems $A = 0$. The investigation of such systems is simple to carry out within the accuracy indicated: $\rho_i = \text{const}$, $\varphi_i = -\beta_i t + \varphi_{i0}$ ($i = 1, 2, \ldots, n$).

From this and from (1.10), for these "quasi-oscillators " (*) we have

$$\rho_j = J_j, \qquad \varphi_j = -\beta_j t + \varphi_{j0} \qquad (j = 4, 5, \ldots, n) \tag{1.13}$$

where φ_{j0} is the initial value of phase φ_j. Thus, relations (1.13) completely determine (in the approximation being considered) the motion of the quasi-oscillators $(\rho_j, \varphi_j) = (j = 4, 5, \ldots, n)$, which has been "constructed" in the variables ρ_j, φ_j as the motion of a point on a circle of constant radius $\rho_j = J_j$ with a constant angular velocity $-\beta_j$ equal to the frequency of the j th oscillator when it is not coupled to the other oscillators. In the variables ρ_α, φ_α the study of the motion of the system (1.9) of oscillators being considered can be carried on independently for the quasi-oscillators (ρ_i, φ_i) $(i = 1, 2, 3)$ "essentially" coupled by the resonance relation (1.4) (i.e., $k_i \neq 0$ $(i = 1, 2, 3)$) and for the remaining quasi-oscillators (ρ_j, φ_j) $(j = 4, 5, \ldots, n)$, whose motion is completely determined by relations (1.13). Therefore, in what follows we shall consider the motion only of the first three quasi-oscillators (ρ_i, φ_i) $(i = 1, 2, 3)$.

2. Equations of motion of "resonant" oscillators. Integrals. Those oscillators whose frequencies enter into the resonance relation (1.4) with a coefficient $k_\alpha \neq 0$ are called "resonant" oscillators. (In our case such are the first three oscillators, or after passing to the normal form of the Hamiltonian, the first three quasi-oscillators (ρ_i, φ_i) $(i = 1, 2, 3)$). It is convenient to rewrite the integrals (1.10) relating to these three quasi-oscillators in the following form, introducing a new notation for the integrals :

$$\rho_1 + \rho_3 = J_3 \equiv I_1, \qquad \rho_2 + \rho_3 = J_2 + J_3 \equiv I_2 \tag{2.1}$$

We write out the equation for ρ_3 ,

$$\frac{d\rho_3}{dt} = 2A\sqrt{\rho_1\rho_2\rho_3}\sin\psi \tag{2.2}$$

If an expression for $\sin\psi$ is obtained with the aid of integral F in (1.11), while ρ_1 and ρ_2 are expressed in terms of ρ_3 with the aid of integrals (2.1), and if these expressions are substituted into (2.2), then for ρ_3 we obtain the following autonomous equation:

$$\frac{d\rho_3}{dt} = \pm 2A\sqrt{\rho_3(I_1-\rho_3)(I_2-\rho_3) - F_1^2} \qquad (F_1 = F/2A) \tag{2.3}$$

The investigation of this equation permits us to obtain a qualitative picture of the possible motions for various initial conditions. Equation (2.3) can be integrated directly for actual values of the initial conditions. The quantities ρ_1 and ρ_2 are obtained then from the integrals I_1 and I_2 in (2.1). The magnitudes of the phases φ_1, φ_2, φ_3 are obtained after this by quadratures from the equations for the phases and, moreover, using integral F, these equations can be written as follows:

$$\begin{gathered}\frac{d\varphi_1}{dt} = -\frac{\partial\Gamma}{\partial\rho_1} = -(\beta_1 + A\sqrt{\rho_2\rho_3/\rho_1}\cos\psi) = -\left(\beta_1 + \frac{F}{2\rho_1}\right)\\ \frac{d\varphi_i}{dt} = -\left(\beta_i + \frac{F}{2\rho_i}\right) \qquad (i = 2, 3)\end{gathered} \tag{2.4}$$

---

*) The pair of variables $(\rho_\alpha, \varphi_\alpha)$ is here called a "quasi-oscillator".

Hence, among other things, we see that the rate of change of the phase φ_i ($i = 1, 2, 3$) depends only on its "own" variable ρ_i, i.e. after the value of ρ_i has been obtained, the equation for φ_i can be integrated independently of the other values of ρ_α, φ_α ($\alpha \neq i$).

3. "Phase picture" of the system in the case $I_1 \neq I_2$. We go on to the investigation of Eq. (2.3). We denote the radicand by $\Phi(\rho_3; F_1)$, i.e.,

$$\Phi(\rho_3;\ F_1) = \rho_3 (I_1 - \rho_3)(I_2 - \rho_3) - F_1^2 \tag{3.1}$$

We investigate this function for fixed values of integrals I_1 and I_2 depending on the value of integral F_1. We first consider the more general case when $I_1 \neq I_2$ and, for definiteness, we let

$$0 < I_1 < I_2 \tag{3.2}$$

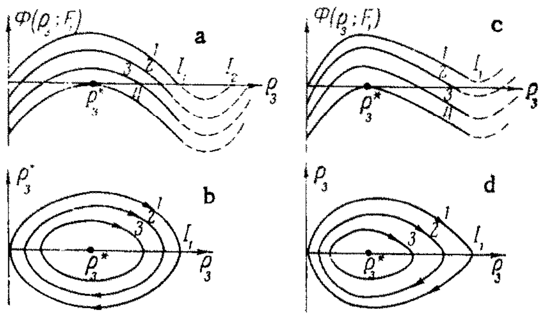

Fig. 1.

(Note that because the ρ_i are nonnegative, the integrals $I_1, I_2 \geqslant 0$.) From expressions (2.1), (3.2) and from the nonnegativeness of the ρ_i it follows that during the whole time of motion,

$$\rho_1(t) \leqslant I_1, \qquad \rho_2(t) \leqslant I_2,$$
$$\rho_3(t) \leqslant I_1 \tag{3.3}$$

The very writing of the cubic polynomial $\Phi(\rho_3; F_1)$ easily enables us to represent it in the form of function (3.1) for $F_1 = 0$ (curve 1 in Fig. 1 a) and consequently, for $F_1 \neq 0$, since curve 1 in Fig. 1 a simply drops by the amount F_1^2. It is not difficult to show that for all possible values of integral F_1 the functions $\Phi(\rho_3; F_1)$ have a maximum at the point

$$\rho_3^* = {}^1/_3\,[I_1 + I_2 - \sqrt{(I_1 + I_2)^2 - 3I_1 I_2}] \tag{3.4}$$

Here ρ_3^* is that root of the equation $\Phi'(\rho_3; F_1) = 0$ which lies in the region of possible motions $0 \leqslant \rho_3 \leqslant I_1$. We note also that $\Phi(\rho_3^*, F_1)$ vanishes when $|\cos\psi| = 1$, or, in other words, for the largest possible value of the integral F_1^2, equal to

$$(F_1^2)^* = (I_1 - \rho_3^*)(I_2 - \rho_3^*)\rho_3^* \tag{3.5}$$

The form of the curves of $\Phi(\rho_3; F_1)$ for fixed values of integrals I_1 and I_2 and for various possible values of integral F_1 are shown in Fig. 1 a. Curve 1 is obtained for $F_1 = 0$, curve 4, for when F_1^2 reaches value (3.5). The remaining curves correspond to intermediate values of integral F_1. The corresponding curves on the phase plane (ρ_3, $\rho_3^{\cdot}$) are shown in Fig. 1 b. There is one singular point

$$\rho_3 = \rho_3^*, \qquad \rho_3^{\cdot} = 0 \tag{3.6}$$

where ρ_3^* is determined by expression (3.4). Integral curves not passing through this singular point are cycles intersecting the ρ_3-axis at right angle.

For all possible initial values, except $\rho_{30} = \rho_3^*$ and simultaneously $|\cos\psi_0| = 1$, there occurs a periodic variation of ρ_3 (and, consequently, from integrals I_1 and I_2, a periodic variation of ρ_1 and ρ_2) which, by taking into account that in the first approximation ρ_i is proportional (with coefficient β_i) to the energy of the ith oscillator and by following the terminology in [5], is called the pumping of energy between the oscillators. The pumping takes place with a period

$$\tau = \oint \frac{d\rho_3}{2A\sqrt{\rho_3(I_1-\rho_3)(I_2-\rho_3)-F_1^2}} \tag{3.7}$$

where the integral is taken along the cycle corresponding to the initial values being considered. Thus, for initial values other than singular point (3.6), there arises a periodic variation of ρ_3 between ρ_{31} and ρ_{32}. The values ρ_{31} and ρ_{32} are the roots of the equation $\Phi(\rho_3;\ F_1) = 0$, lying on the segment $0 \leqslant \rho_3 \leqslant I_1$. The more the value of the integral F_1^2 differs from its largest possible value (3.5) the greater the limits within which the quantity ρ_3 (consequently, ρ_1 and ρ_2) varies and the "deeper" [5] the pumping of the energy.

In the case $I_1 < I_2$ the curve *1* in Fig. 1 b corresponds to a periodic pumping mode of the energy and, moreover, on this curve there takes place a "complete" pumping of energy between the third and the other two oscillators, i.e., ρ_3 oscillated periodically between the values $\rho_3 = 0$ and $\rho_3 = I_1$. Here ρ_1 oscillates periodically between $\rho_1 = I_1$ and $\rho_1 = 0$, while ρ_2 oscillates periodically between the values I_2 and $(I_2 - I_1)$. The period of these oscillations is given by formula (3.7) if as the cycle we take curve *1* of Fig. 1 b, and take into account that here $F_1 = 0$, i.e.,

$$\tau = \int_0^{I_1} \frac{d\rho_3}{A\sqrt{\rho_3(I_1-\rho_3)(I_2-\rho_3)}}$$

This quantity is finite if $I_1 \neq I_2$ We remark that the motion of the system being considered takes place in such a way that when ρ_3 takes a maximal (or a minimal) value the quantities ρ_1 and ρ_2 simultaneously take minimal (or maximal) values, respectively.

4. "Phase portrait" in the case $I_1 = I_2$ This case is of interest in that a limit mode arises. The equation for ρ_3 takes the form

$$\rho_3^{\cdot} = \pm 2A\sqrt{\rho_3(I_1-\rho_3)^2 - F_1^2} \tag{4.1}$$

The form of the curves

$$\Phi(\rho_3;\ F_1) = \rho_3(I_1-\rho_3)^2 - F_1^2 \tag{4.2}$$

is shown in Fig. 1c. Curve *1* of Fig. 1c corresponding to the case $F_1 = 0$, is tangent to the ρ_3-axis at the point $\rho_3 = I_1$. The function $\Phi(\rho_3;\ F_1)$ reaches a maximum at the point

$$\rho_3 = \rho_3^* = I_1 / 3 \tag{4.3}$$

on the considered interval $0 \leqslant \rho_3 \leqslant I_1$ for any possible value of F_1.

For the maximal value of

$$F_1^2 = \rho_3^*(I_3 - \rho_3^*)^2 = {}^4/_{27}\, I_1^3$$

the curve of $\Phi(\rho_3;\ F_1)$ has the form of curve *4* in Fig. 1 c. The remaining curves in Fig. 1 c correspond to intermediate values of F_1^2. The corresponding curves on the phase plane $(\rho_3,\ \rho_3^{\cdot})$ are shown in Fig. 1 d. In the case being considered there are two singular points:

$$\rho_3 = \rho_3^* = {}^1/_3 I_1, \qquad \rho_3^{\cdot} = 0 \tag{4.4}$$

$$\rho_3 = I_1, \qquad \rho_3^{\cdot} = 0 \tag{4.5}$$

The basic qualitative difference between Fig. 1 d and Fig. 1 b, which latter corresponds to the case $I_1 < I_2$, is the appearance of the singular point (4.5) and, correspondingly, of a separatrix (curve *1* in Fig. 1 d). All curves on the phase plane, not passing through the singular points, are cycles (Fig. 1 d). These cycles correspond to a periodic pumping of energy between the oscillators, with a period

$$\tau = \oint \frac{d\rho_3}{2A\sqrt{(I_1-\rho_3)^2 - F_1^2}}$$

For the initial values corresponding to the separatrix ($F_1 = 0$) there arises a limit motion. The representative point (Fig. 1 d) takes infinite time to go into the point (4.5): for $F_1 = 0$ integral (4.6) has a singularity at the point $\rho_3 = I_1$ and diverges. Thus, the separatrix corresponds, in our case of $I_1 = I_2$ to the mode of complete pumping ("transmission") of the energy of the two quasi-oscillators (ρ_1, φ_1) and (ρ_2, φ_2) into the "fast" quasi-oscillator (ρ_3, φ_3) (recall that $\beta_3 = \beta_1 + \beta_3$). This pumping lasts infinitely long.

5. Periodic motions. A periodic motion of the oscillators occurs for initial values corresponding to a center-type singular point, both in the case $I_1 < I_2$ as well as in the case $I_1 = I_2$. We first consider the periodic motion in the case $I_1 < I_2$.

At a center-type point ($\rho_3 = \rho_3^*$, $\dot\rho_3 = 0$) the value ρ_3^* is given by expression (3.4) and the value of the resonance phase ψ is such that $|\cos\psi| = 1$; therefore, from Eqs. (2.4) we see that

$$d\varphi_3 / dt = -(\beta_3 \pm A\sqrt{\rho_1^*\rho_2^* / \rho_3^*}) \tag{5.1}$$
$$\rho_1^* = I_1 - \rho_3^*, \qquad \rho_2^* = I_2 - \rho_3^*$$

The sign in front of A is chosen depending on the sign of $\cos\psi$ (equal, in our case, to +1 or -1). Analogous equations for φ_1 and φ_2 are easily obtained from (2.4). Thus, to a center-type point there correspond two types of periodic motions:

$$\begin{aligned}
&\rho_1(t) = \rho_1^*, \qquad \varphi_1(t) = -(\beta_1 \pm A\sqrt{\rho_2^*\rho_3^* / \rho_1^*})\,t + \varphi_{10}\\
&\rho_2(t) = \rho_2^*, \qquad \varphi_2(t) = -(\beta_2 \pm A\sqrt{\rho_1^*\rho_3^* / \rho_2^*})\,t + \varphi_{20}\\
&\rho_3(t) = \rho_3^*, \qquad \varphi_3(t) = -(\beta_3 \pm A\sqrt{\rho_1^*\rho_2^*/\rho_3^*})\,t + \varphi_{30}
\end{aligned} \tag{5.2}$$

Here $\varphi_{10} + \varphi_{20} - \varphi_{30} = \psi_0$ and, moreover, $\cos\psi_0 = 1$ for the first type of periodic motion and $\cos\psi_0 = -1$ for the second type. The motion of the representative point in the coordinates (ρ_3, φ_3) is the motion of a point on a circle of radius $\rho_3 = \rho_3^*$ with constant angular velocity $-(\beta_3 + A\sqrt{\rho_2^*\rho_3^*/\rho_1^*})$ for one type of periodic motion and angular velocity $-(\beta_3 - A\sqrt{\rho_2^*\rho_3^*/\rho_1^*})$ for the other type. An analogous portrait exists in the planes (ρ_1, φ_1), (ρ_2, φ_2). Note that the frequencies in the two different types of motion being considered is in one case larger, and in the other case smaller, than the natural frequency of the corresponding oscillator. In these periodic motions the nonlinear coupling between the oscillators manifests itself in the alteration in the frequency of their oscillations, whereas the quantities $\rho_i = \rho_i^*$ ($i = 1, 2, 3$) are preserved, i.e., there is no pumping of energy.

In the case $I_1 = I_2$ the frequencies of the periodic motions corresponding to a center are simpler in form and, therefore, the periodic solutions (5.2) take the simpler form

$$\begin{aligned}
&\rho_1(t) = {}^2/_3\, I_1, \qquad \varphi_1(t) = -(\beta_1 \pm A\sqrt{{}^1/_3\, I_1})\,t + \varphi_{10}\\
&\rho_2(t) = {}^2/_3\, I_1, \qquad \varphi_2(t) = -(\beta_2 \pm A\sqrt{{}^1/_3\, I_1})\,t + \varphi_{20}\\
&\rho_3(t) = {}^1/_3\, I_1, \qquad \varphi_3(t) = -(\beta_3 \pm 2A\sqrt{{}^1/_3\, I_1})\,t + \varphi_{30}\\
&\varphi_{10} + \varphi_{20} - \varphi_{30} = \psi_0, \qquad \cos\psi_0 = +1 \text{ or } -1
\end{aligned} \tag{5.3}$$

For small deviations from the initial conditions corresponding to a center, both in the case $I_1 < I_2$ (Fig. 1 b) as well as in the case $I_1 = I_2$ (Fig. 1 d), the representative point describes small circles around the center, i.e., the quantities ρ_i $(i = 1, 2, 3)$ perform small periodic oscillations around ρ_i^*. In the case $I_1 = I_2$ a periodic motion corresponds also to the singular point $(\rho_3 = I_1,\ \rho_3^{\cdot} = 0)$ for which only the one "fast" quasi-oscillator

$$\rho_3 = I_1,\ \ \varphi_3(t) = -\beta_3 t + \varphi_{30}$$

"moves". However, the nature of this periodic motion is such that for the least change in the initial conditions the representative point (Fig. 1 d) starts to move along a cycle close to the separatrix, which corresponds to a "slow" pumping of energy between the oscillators.

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ON THE RELATION BETWEEN RADIAL AND VERTICAL OSCILLATIONS OF PARTICLES IN CYCLOTRONS

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We consider the betatron oscillations of particles in cyclotrons with weak focusing. The equations of motion of the particles are described in the form of a fourth-order Liapunov system [1, 2]. On the basis of a transformation of Liapunov systems, proposed by the author [3, 4], the equations of motion are reduced to a second-order nonautonomous equation containing a small parameter. The vertical-radial oscillations of the particles are determined with the aid of the